# THE DISTRIBUTION OF RATIONALS IN RESIDUE CLASSES

CRISTIAN COBELI AND ALEXANDRU ZAHARESCU

Abstract. Our purpose is to give an account of the $r$-tuple problem on the increasing sequence of reduced fractions having denominators bounded by a certain size and belonging to a fixed real interval. We show that when the size grows to infinity, the proportion of the $r$-tuples of consecutive denominators with components in certain apriori fixed arithmetic progressions with the same ratio approaches a limit, which is independent on the interval. The limit is given explicitly and it is completely described in a few particular instances.

## 1. Introduction

Let $Q$ be a positive integer and let $\mathcal{I}$ be an interval of real numbers. We denote by $\mathfrak{F}_Q^{\mathcal{I}}$ the sequence of reduced fractions from $\mathcal{I}$, whose denominators are positive and $\leq Q$. The elements of the sequence are assumed to be arranged in ascending order. Since the denominators of these fractions are periodic with respect to an unit interval, and they also determine uniquely the numerators, one usually focuses on $\mathfrak{F}_Q$, the sequences corresponding to $\mathcal{I} = [0, 1]$. This is known as the Farey sequence of order $Q$.

Questions concerned with Farey sequences have a long history. In some problems, such as those related to the connection between Farey fractions and Dirichlet $L-$functions, one is lead to consider subsequences of Farey fractions defined by congruence constraints. Knowledge of the distribution of subsets of Farey fractions with congruence constraints would also be useful in the study of the periodic two-dimensional Lorentz gas. This is a billiard system on the two-dimensional torus with one or more circular regions (scatterers) removed (see Sinaĭ [24], Bunimovich [8], Chernov [9], Boca and Zaharescu [7]). Such systems were introduced in 1905 by Lorentz [23] to describe the dynamics of electrons in metals. A problem raised by Sinaĭ on the distribution of the free path length for this billiard system, when small scatterers are placed at integer points and the trajectory of the

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particle starts at the origin, was solved in Boca et all [5], [6], using techniques developed in [1], [2], [3] to study the local spacing distribution of Farey sequences.

The more general case when the trajectory starts at a given point with rational coordinates is intrinsically connected with the problem of the distribution of Farey fractions satisfying congruence constraints. For example, the case when the trajectory starts from the center $(1/2, 1/2)$ of the unit square is related to the distribution of Farey fractions with odd numerators and denominators.

Some questions on the distribution of Farey fractions with odd, and respectively even denominators have been investigated in [4], [10], [12], and [22]. Most recently, the authors [13] have proved the existence of a density function of neighbor denominators that are in an arbitrary arithmetic progression.

Let $d \geq 2$ and $0 \leq c \leq d-1$ be integers. We denote by $\mathfrak{F}_Q(c,d)$ the set of Farey fractions of order $Q$ with denominators $\equiv c \pmod{d}$. We assume that the elements of $\mathfrak{F}_Q(c,d)$ are arranged increasingly. In various problems on the distribution of Farey fractions with denominators in a certain arithmetic progression it would be very useful to understand how the set $\mathfrak{F}_Q(c,d)$ sits inside $\mathfrak{F}_Q$. With this in mind, in the present paper we investigate the distribution modulo $\mathfrak{d}$ of the components of tuples of consecutive Farey fractions. To make things precise, let $\mathbf{c} = (c_1, \dots, c_r) \in [0, \mathfrak{d}-1]^r \cap \mathbb{N}^r$, for some integer $r \geq 1$. We say that a tuple $\mathbf{q} = (q_1, \dots, q_r)$ of consecutive denominators in $\mathfrak{F}_Q$ has *parity* $\mathbf{c}$ if $q_j \equiv c_j \pmod{\mathfrak{d}}$, for $1 \leq j \leq r$. In this case, we write shortly $\mathbf{q} \equiv \mathbf{c} \pmod{\mathfrak{d}}$. We also say that $\mathbf{c}$ is the *parity* of the tuple of fractions $\mathbf{f} = (a_1/q_1, \dots, a_r/q_r) \in \mathfrak{F}_Q^r$. The question, we address here, is whether it is true that $\rho_Q(\mathbf{c}, \mathfrak{d})$, the proportion of $r-$tuples of consecutive Farey fractions $\mathbf{f} \in \mathfrak{F}_Q^r$ of parity $\mathbf{c}$, has a limit $\rho(\mathbf{c}, \mathfrak{d})$, as $Q \to \infty$. And, if so, can one provide an explicit formula for $\rho(\mathbf{c}, \mathfrak{d})$? More generally, the same questions can be asked for the subset of Farey fractions which belong to a given real interval.

These problems have positive answers, indeed. Thus, for any interval of positive length $\mathcal{I}$, any $r \geq 1$ and any choice of parities $\mathbf{c}$, the proportion of $r-$tuples of consecutive fractions in $\mathfrak{F}_Q^{\mathcal{I}}$ of parity $\mathbf{c}$ approaches a limit as $Q \to \infty$. Moreover, this limit depends on $r$, $\mathbf{c}$ and $\mathfrak{d}$ only, and is independent of the choice of the interval $\mathcal{I}$. Roughly, this says that the probability that $r$ consecutive fractions have parity $\mathbf{c}$ is independent of the position of these fractions in $\mathbb{R}$.

For instance, when $r = 1$, one finds that the probability that a randomly chosen fraction is even[a] equals 1/3. In other words, there are asymptotically twice as many odd fractions as even fractions in $\mathfrak{F}_Q$. For $r \geq 2$, the parities of $r$ consecutive fractions are not independent of one another. Thus, for instance, the probability that two neighbor fractions are even is zero. This follows by a classical fundamental property, which says that any consecutive Farey fractions $a'/q', a''/q''$ satisfy

$$a''q' - a'q'' = 1. \tag{1}$$

This implies that $q', q''$ are relatively prime, so not both of them are even.

In Section 2 we present some properties of $\mathfrak{F}_Q$ that will be used in the proofs. The next sections are devoted to give precise statements of the main results and their proofs. We conclude with a few examples for some special arithmetic progressions.

## 2. Facts about $\mathfrak{F}_Q$

Here we state a few fact about Farey sequences that are needed in the sequel. For the proofs and in depth details we refer to [2], [3], [15], [17] and [20].

For any $r \geq 1$, let $\mathfrak{F}_Q^r$ and $\mathfrak{D}_Q^r$ be the set of $r$-tuples of consecutive fractions in $\mathfrak{F}_Q$, respectively denominators of fractions in $\mathfrak{F}_Q$.

We begin with a geometric correspondent of $\mathfrak{F}_Q$. Let $\mathcal{T}$ be the triangle with vertices $(1,0)$, $(1,1)$, $(0,1)$ and its scaled transform, $\mathcal{T}_Q := Q\mathcal{T}$. Any lattice point with coprime coordinates $(q', q'') \in Q\mathcal{T}$ determines uniquely a pair of consecutive fractions $(a'/q', a''/q'') \in \mathfrak{F}_Q$, and conversely, if $(a'/q', a''/q'') \in \mathfrak{F}_Q$ then $\gcd(q', q'') = 1$, $q' + q'' > Q$, so $(q', q'') \in Q\mathcal{T}$.

Next, let us notice that one way to read (1) yields $a' = -\overline{q''} \pmod{q'}$ and $a' = \overline{q'} \pmod{q'}$. (Here the representative of $\overline{x}$, the inverse of $x \pmod{d}$, is taken in the interval $[0, d-1]$.) This is the reason and the main idea which supports the fact many properties regarding the distribution of $\mathfrak{F}_Q$ are preserved on $\mathfrak{F}_Q \cap \mathcal{I}$, for any $\mathcal{I} \subseteq [0,1]$, and then by periodicity for all larger intervals $\mathcal{I}$.

Another noteworthy property says that, starting with a pairs $(q', q'') \in \mathfrak{D}_Q^2$, then one may calculate by a recursive method all the fractions that precede or succeed $a'/q'$ and $a''/q''$. To see this, we first remark that if $a'/q'$ and $a''/q''$ are not neighbor in $\mathfrak{F}_Q$, then (1) no longer holds true. However, there is a good replacement. Indeed, if

[a]We call a fraction *odd* or *even* according to whether its denominator is odd or even.

$(a'/q', a''/q'', a'''/q''') \in \mathfrak{F}_Q^3$, then the *median* fraction is linked to the extremes through the integer

$$k = \frac{a' + a'''}{a''} = \frac{q' + q'''}{q''} = \left[\frac{q' + Q}{q''}\right]. \tag{2}$$

Suppose now that $r \geq 3$ and $\mathbf{q} = (q_1, \dots, q_r) \in \mathfrak{D}_Q^r$, and consider the positive integers defined by

$$k_j = \frac{q_j + q_{j+2}}{q_{j+1}} = \left[\frac{q_j + Q}{q_{j+1}}\right] \quad \text{for } j = 1, \dots, r-2\,.$$

Then we may say that $(q_1, q_2)$ *generates* $\mathbf{q}$ and $\mathbf{k} = (k_1, \dots, k_{r-2})$, too. In order to emphasize this, we write

$$\mathbf{q}(q_1, q_2) = \mathbf{q}_Q^r(q_1, q_2) = (q_1, \dots, q_r),$$
$$\mathbf{k}(q_1, q_2) = \mathbf{k}_Q^{r-2}(q_1, q_2) = (k_1, \dots, k_{r-2}),$$

dropping the sub- and superscript when they are clear from the context.

For a given positive integer $s$, let $\mathcal{A}^s$ be the set of all *admissible* $\mathbf{k} \in (\mathbb{N}^*)^s$, which we define to be those vectors $\mathbf{k}$ generated by some $(q', q'') \in \mathcal{T}_Q$, for some $Q \geq 1$ with $\gcd(q', q'') = 1$. We remark that $\mathcal{A}^s \subsetneq (\mathbb{N}^*)^s$, and the components of vectors from $\mathcal{A}^s$ should relate to one another in specific manners, such as: a neighbor of 1 can be any integer $\geq 2$, neighbors of 2 can be only $1, 2, 3$ or 4, neighbors of 3 can be only 1 or 2, neighbors of 4 can be only 1 or 2, and a neighbor of any $k \geq 5$ can be only 1. Though, these conditions do not characterize completely the elements of $\mathcal{A}^s$ for $s \geq 3$, and the complexity grows with $r$.

Although $\mathbf{q}(q_1, q_2)$ is uniquely generated by its first two components $q_1, q_2$, the vectors $\mathbf{k} \in \mathcal{A}^s$ have each infinitely many generators. The set of generators of a $\mathbf{k} \in \mathcal{A}^s$ can be nicely described and it will play a special role in what follows. We do this by introducing its natural continuum envelope downscaled by a factor of $Q$. Let us see the precise definitions. For any $(x, y) \in \mathbb{R}^2$, let $\big\{L_j(x, y)\big\}_{j \geq -1}$ be the sequence given by $L_{-1}(x, y) = x$, $L_0(x, y) = y$ and recursively, for $j \geq 1$,

$$L_j(x, y) = \left[\frac{1 + L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] L_{i-1}(x, y) - L_{i-2}(x, y)\,.$$

Then, we put

$$\mathbf{k} : \mathcal{T} \to (\mathbb{N}^*)^s, \qquad \mathbf{k}(x, y) = \big(k_1(x, y), \dots, k_s(x, y)\big),$$

where, for $1 \leq j \leq s$,

$$k_j(x, y) = \left[\frac{1 + L_{j-2}(x, y)}{L_{j-1}(x, y)}\right]. \tag{3}$$

Now, for any $\mathbf{k} \in (\mathbb{N}^*)^s$, we define

$$\mathcal{T}[\mathbf{k}] = \mathcal{T}^s[\mathbf{k}] := \left\{(x,y) \in \mathcal{T}:\ \mathbf{k}(x,y) = \mathbf{k}\right\},$$

the domain on which the map $\mathbf{k}(x,y)$ is locally constant. The definition may be extended for an empty $\mathbf{k}$ (with no components) by putting $\mathcal{T}^0[\cdot] = \mathcal{T}$, the Farey triangle. It turns out that $\mathcal{T}[\mathbf{k}]$ is always a convex polygon. Also, for any fixed $s \geq 0$, the set of all polygons $\mathcal{T}[\mathbf{k}]$, with $\mathbf{k} \in \mathcal{A}^s$, form partition of $\mathcal{T}$, that is, $\mathcal{T} = \bigcup\limits_{\mathbf{k}\in\mathcal{A}^s} \mathcal{T}[\mathbf{k}]$ and $\mathcal{T}[\mathbf{k}] \cap \mathcal{T}[\mathbf{k}'] = \emptyset$, whenever $\mathbf{k}, \mathbf{k}' \in \mathcal{A}^s$, $\mathbf{k} \neq \mathbf{k}'$.

The set we were looking for is the polygon $Q\mathcal{T}[\mathbf{k}]$, which contains all lattice points from $Q\mathcal{T}$ with relatively prime coordinates that generate $\mathbf{k}$.

We conclude this section by mentioning that the symmetric role played by numerators and denominators in (1) assures that just about any statement, such as Theorems 1-4 below, holds identically with the word 'denominator' replaced by the word 'numerator'.

## 3. The finite probability $\rho^r_Q(\mathbf{c}, \mathfrak{d})$

Let $\mathbf{N}^r_Q(\mathbf{c}, \mathfrak{d})$ be the number of r-tuples of denominators of consecutive fractions in $\mathfrak{F}_Q$ that are congruent with $\mathbf{c}$ modulo $\mathfrak{d}$, that is $\mathbf{N}^r_Q(\mathbf{c}, \mathfrak{d})$ is the cardinality of the set

$$\mathfrak{G}^r_Q(\mathbf{c}, \mathfrak{d}) = \left\{(q_1, \ldots, q_r) \in \mathfrak{D}^r_Q:\ q_j \equiv c_j \pmod{\mathfrak{d}},\ \text{for } j = 1, \ldots, r\right\}.$$

Then we ought to to find the proportion

$$\rho^r_Q(\mathbf{c}, \mathfrak{d}) := \frac{\mathbf{N}^r_Q(\mathbf{c}, \mathfrak{d})}{\#\mathfrak{F}_Q - (r-1)}$$

and probe the existence of a limit of $\rho^r_Q(\mathbf{c}, \mathfrak{d})$, as $Q \to \infty$.

In order to find $\mathbf{N}^r_Q(\mathbf{c}, \mathfrak{d})$, we need to estimate the number of points with integer coordinates having a certain parity, and belonging to different domains. This is the object of the next section.

## 4. Lattice Points in Plane Domains

Given a set $\Omega \subset \mathbb{R}^2$ and integers $0 \leq a, b < \mathfrak{d}$, let $N'_{a,b;\mathfrak{d}}(\Omega)$ be the number of lattice points in $\Omega$ with relatively prime coordinates congruent modulo $\mathfrak{d}$ to $a$ and $b$, respectively, that is,

$$N'_{a,b;\mathfrak{d}}(\Omega) = \left\{(m,n) \in \Omega:\ m \equiv a \pmod{\mathfrak{d}};\ n \equiv b \pmod{\mathfrak{d}};\ \gcd(m,n) = 1\right\}.$$

Notice that $N'_{a,b;\mathfrak{d}} = 0$ if $\gcd(a,b) > 1$, so we may assume that $a, b$ are relatively prime.

**Lemma 1.** *Let $R > 0$ and let $\Omega \subset \mathbb{R}^2$ be a convex set of diameter $\le R$. Let $\mathfrak{d}$ be a positive integer and let $0 \le a, b < \mathfrak{d}$, with $\gcd(a,b) = 1$. Then*

$$N'_{a,b;\mathfrak{d}}(\Omega) = \frac{6}{\pi^2\mathfrak{d}^2} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{p^2}\right)^{-1} \operatorname{Area}(\Omega) + O\big(R\log R\big) . \tag{4}$$

*Proof.* As $\mathfrak{d}$ is fixed and $R$ can be taken large enough, we may assume that $\Omega \subset [0,R]^2$. Removing the coprimality condition by Möbius summation, we have:

$$N'_{a,b;\mathfrak{d}}(\Omega) = \sum_{\substack{(m,n)\in\Omega \\ m\equiv a \,(\mathrm{mod}\ \mathfrak{d}) \\ n\equiv b\,(\mathrm{mod}\ \mathfrak{d}) \\ \gcd(m,n)=1}} 1 = \sum_{\substack{(m,n)\in\Omega \\ m\equiv a \,(\mathrm{mod}\ \mathfrak{d}) \\ n\equiv b\,(\mathrm{mod}\ \mathfrak{d})}} \sum_{\substack{e|m \\ e|n}} \mu(e) = \sum_{e=1}^{R} \mu(e) \sum_{\substack{(m,n)\in\Omega \\ m\equiv a \,(\mathrm{mod}\ \mathfrak{d}) \\ n\equiv b\,(\mathrm{mod}\ \mathfrak{d}) \\ e|m,\ e|n}} 1 \,.$$

In the last sum, the last four conditions can be rewritten as: $m = em_1$, $n = en_1$, $em_1 \equiv a \pmod{\mathfrak{d}}$, $en_1 \equiv b \pmod{\mathfrak{d}}$, for some integers $m_1, n_1$. Since $\gcd(a,b) = 1$, it follows that $(e,\mathfrak{d}) = 1$. Let $e^{-1}$ be the inverse of $e \bmod \mathfrak{d}$. Then,

$$N'_{a,b;\mathfrak{d}}(\Omega) = \sum_{1\le e\le R} \mu(e) \sum_{\substack{(m_1,n_1)\in\frac{1}{e}\Omega \\ m_1\equiv e^{-1}a \,(\mathrm{mod}\ \mathfrak{d}) \\ n_1\equiv e^{-1}b\,(\mathrm{mod}\ \mathfrak{d})}} 1 \,.$$

Here the inner sum is equal to $\frac{1}{\mathfrak{d}^2}\operatorname{Area}\big(\frac{1}{e}\Omega\big) + O\Big(\operatorname{length}\Big(\partial\big(\frac{1}{e}\Omega\big)\Big)\Big)$, therefore

$$N'_{a,b;\mathfrak{d}}(\Omega) = \frac{1}{\mathfrak{d}^2}\operatorname{Area}(\Omega) \sum_{\substack{1\le e\le R \\ \gcd(e,\mathfrak{d})=1}} \frac{\mu(e)}{e^2} + O\big(R\log R\big) \,.$$

Completing the last sum and the Euler product for the Dirichlet series, we have

$$\sum_{\substack{1\le e\le R \\ \gcd(e,\mathfrak{d})=1}} \frac{\mu(e)}{e^2} = \sum_{\substack{1\le e\le \infty \\ \gcd(e,\mathfrak{d})=1}} \frac{\mu(e)}{e^2} + O(R^{-1}) = \prod_{p} \left(1 - \frac{1}{p^2}\right) \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{p^2}\right)^{-1} + O(R^{-1}) \,.$$

This completes the proof of the lemma, since the first product is equal to $6/\pi^2$. □

In particular, Lemma 1 provides an estimation of the cardinality of $\mathfrak{F}_Q(c,d)$.

**Lemma 2.** *Let $d \ge 1$ and $0 \le c \le d-1$ be integers. Then*

$$\#\mathfrak{F}_Q(c,d) = \frac{3\nu(c,d)}{\pi^2\mathfrak{d}^2} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{p^2}\right)^{-1} Q^2 + O\big(Q\log Q\big) \,,$$

*where $\nu(c,d) = \frac{\varphi(c)}{c} d + O(c)$ is the number of positive integers $\le d$ that are relatively prime with $c$.*

*Proof.* Let $\Omega = \mathcal{T}_Q$ be the triangle with vertices $(Q,0)$, $(Q,Q)$, $(0,Q)$. Then, via what we know from Section 2, counting only by the first component of a pair of consecutive elements of $\mathfrak{F}_Q$, we find that $\#\mathfrak{F}_Q(c,d)$ is the number of lattice points $(a,b) \in \mathcal{T}_Q$ with $\gcd(a,b)=1$, $a \equiv c \pmod d$ and no other condition on $b$. Then the proof follows, since by Lemma 1, we get

$$\begin{aligned}\#\mathfrak{F}_Q(c,d) &= \sum_{\substack{b=1\\ \gcd(b,d)=1}}^{d} N'_{c,b;\mathfrak{d}}(\mathcal{T}_Q)\\ &= \frac{6}{\pi^2\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{\substack{b=1\\ \gcd(b,d)=1}}^{d}\frac{Q^2}{2}+O\big(Q\log Q\big)\,.\end{aligned}$$

□

In particular, by Lemma 2, we find that $\#\mathfrak{F}_Q = \frac{3}{\pi^2}Q^2+O(Q\log Q)$ and the cardinality of the subsets of fractions with odd and even denominators are $\#\mathfrak{F}_{Q,\text{odd}} = \frac{2}{\pi^2}Q^2+O(Q\log Q)$ and $\#\mathfrak{F}_{Q,\text{odd}} = \frac{1}{\pi^2}Q^2+O(Q\log Q)$, respectively.

## 5. The densities $\rho^r_Q(\mathbf{c},\mathfrak{d})$ and $\rho^r(\mathbf{c},\mathfrak{d})$

We start with two particular cases. First we let $r=1$. Then $\mathbf{c}=c_1$ and, making use of Lemma 2, we obtain

$$\begin{aligned}\rho^1_Q(c_1,\mathfrak{d}) &= \frac{\mathbf{N}^1_Q(c_1,\mathfrak{d})}{\#\mathfrak{F}_Q} = \frac{\#\mathfrak{F}_Q(c_1,\mathfrak{d})}{\#\mathfrak{F}_Q}\\ &= \frac{\nu(c_1,\mathfrak{d})}{\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1} + O\big(Q^{-1}\log Q\big)\,.\end{aligned} \tag{5}$$

Next let $r=2$. Now the numerator of $\rho^2_Q(\mathbf{c},\mathfrak{d})$ is the number of lattice points from $\mathcal{T}_Q$ with relatively prime coordinates and congruent with $\mathbf{c}=(c_1,c_2)$, also. This is exactly the number counted in Lemma 1, thus we get:

$$\begin{aligned}\rho^2_Q(c_1,c_2) &= \frac{\mathbf{N}^2_Q(c_1,c_2;\mathfrak{d})}{\#\mathfrak{F}_Q-1} = \frac{N'_{c_1,c_2;\mathfrak{d}}(\mathcal{T}_Q)}{\#\mathfrak{F}_Q-1}\\ &= \frac{1}{\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1} + O\big(Q^{-1}\log Q\big)\,.\end{aligned} \tag{6}$$

Now we assume that $r\ge 3$ and $\mathbf{c}=(c_1,\dots,c_r)$. We denote by $\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})$ the set of all $\mathbf{k}=(k_1,\dots,k_{r-2})\in\mathcal{A}^{r-2}$ that correspond to $r$-tuples of consecutive denominators that

are congruent to $\mathbf{c} \pmod{\mathfrak{d}}$, that is

$$\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d}) = \left\{ \mathbf{k}\in\mathcal{A}^{r-2}\colon \begin{array}{l} \exists Q\geq 1,\ \exists(q',q'')\in\mathcal{T}_Q,\ \gcd(q',q'')=1,\\ \mathbf{k}(q',q'')=\mathbf{k},\ \mathbf{q}(q',q'')\equiv\mathbf{c} \pmod{\mathfrak{d}} \end{array} \right\}.$$

The natural continuum envelope of the generators of $\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})$ is the set

$$\begin{aligned}\mathcal{E}_Q(\mathbf{c},\mathfrak{d}) &= \left\{(x,y)\in\mathcal{T}_Q\colon\ \mathbf{k}(x,y)\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})\right\}\\ &= \bigcup_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} Q\mathcal{T}[\mathbf{k}]\,,\end{aligned}$$

the union being disjoint. Here are the lattice points that we have to count:

$$\begin{aligned}\mathcal{G}_Q(\mathbf{c},\mathfrak{d}) &= \left\{(q',q'')\in\mathcal{T}_Q\colon \begin{array}{l}\gcd(q',q'')=1,\ \mathbf{k}(q'/Q,q''/Q)\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d}),\\ q_1\equiv c_1 \pmod{\mathfrak{d}},\ q_2\equiv c_2 \pmod{\mathfrak{d}}\end{array}\right\}\\ &= \bigcup_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} \left\{(q',q'')\in Q\mathcal{T}[\mathbf{k}]\colon\ \gcd(q',q'')=1,\ (q_1,q_2)\equiv(c_1,c_2) \pmod{\mathfrak{d}}\right\}.\end{aligned}$$

Then

$$\mathbf{N}^r_Q(\mathbf{c},\mathfrak{d}) = N'_{c_1,c_2;\mathfrak{d}}(\mathcal{E}_Q(\mathbf{c},\mathfrak{d})) = \sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} N'_{c_1,c_2;\mathfrak{d}}\big(Q\mathcal{T}[\mathbf{k}]\big)\,.$$

Applying Lemma 1, this can be written as

$$\begin{aligned}\mathbf{N}^r_Q(\mathbf{c},\mathfrak{d}) = \frac{6Q^2}{\pi^2\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1} &\sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} \operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big)\\ &+ O\Big(Q\log Q \sideset{}{'}\sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} \operatorname{length}\big(\partial\mathcal{T}[\mathbf{k}]\big)\Big)\,.\end{aligned} \tag{7}$$

The prime attached to the sum in the error term indicates that in the summation are excluded those vectors $\mathbf{k}$ for which $\overline{Q\mathcal{T}[\mathbf{k}]}$, the adherence of $Q\mathcal{T}[\mathbf{k}]$, contains no lattice points. We remark that the same estimate applies when $r=1,2$, as noticed in the beginning of this section. Depending on $r$, $\mathbf{c}$ and $\mathfrak{d}$, the nature of the sums in (7) may be different. In any case the first series is finite, being always bounded from above by $\operatorname{Area}(\mathcal{T})=1/2$. In the second series, most of the times are summed infinitely many terms, but one expects that its rate of convergence is small enough to assure a limit for $\rho^r_Q(\mathbf{c},\mathfrak{d})$ as $Q\to\infty$. We summarize the result in the following theorem.

**Theorem 1.** *Let $r\geq 3$, $\mathfrak{d}\geq 2$ and $0\leq c_1,\dots,c_r\leq\mathfrak{d}-1$ be integers. Then*

$$\rho^r_Q(\mathbf{c},\mathfrak{d}) = \frac{2}{\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})}\operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big) + O\big(L_Q(\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d}))Q^{-1}\log Q\big)\,, \tag{8}$$

*where* $L_Q\big(\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})\big)$ *is the sum of the perimeters of all polygons* $Q\mathcal{T}[\mathbf{k}]$ *with* $\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})$*, and having the property that* $\overline{Q\mathcal{T}[\mathbf{k}]}$ *contains lattice points.*

In order to have a limit of the ratio $\rho_Q^r(\mathbf{c},\mathfrak{d})$, we need at least to know that the limit $L_Q\big(\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})\big)/Q^2\to 0$, as $Q\to\infty$, exists. A few initial checks leads one to expect a much stronger estimate to be true. Indeed, the authors [14] have proved that for any fixed $r\geq 1$, the number of $\mathbf{k}\in\mathcal{A}^r$ with all components $\leq Q$ is $rQ+O(r)$. Moreover, for $Q$ sufficiently large, the polygons corresponding to $\mathbf{k}$ with components larger than $Q$ (and only one components can be so!) form at most two connected regions in $\mathcal{T}$. These are smaller and smaller when $Q\to\infty$ and tend to one limit point, $(1,0)$, in the case $r=1$, and to two limit points, $(1,1)$ and $(1,0)$, when $r\geq 2$. Regarding the perimeters of $\mathcal{T}[\mathbf{k}]$, in [14] and [15] we have shown that

$$\sum_{\substack{\mathbf{k}\in\mathcal{A}^r\\ 1\leq k_1,\dots,k_r\leq Q}} \operatorname{length}\big(\partial\mathcal{T}[\mathbf{k}]\big) \ll r\log Q\,, \tag{9}$$

and consequently this is also an upper bound for $L_Q\big(\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})\big)$. One can find in [15] more details on the tessellation of $\mathcal{T}$ formed by the polygons $\mathcal{T}[\mathbf{k}]$ with $\mathbf{k}$ of the same order $r$. For instance, the polygons $\mathcal{T}[\mathbf{k}]$, whose vectors are excepted in the domain of summation in (9) have the form $\mathbf{k}=(\underbrace{2,\dots,2,1}_{s\text{ components}},k,\underbrace{1,2,\dots,2}_{t\text{ components}})$, with $s,t\geq 0$, $s+1+t=r$ and $k>Q$, are quadrangles whose vertices are given by formulas that are calculated explicitly. These are exactly the polygons that have the main contribution (by their number) in the estimate (9), since the number of the remaining ones is always finite. Employing these information in Theorem 1, we obtain our main result.

**Theorem 2.** *Let* $r\geq 3$*,* $\mathfrak{d}\geq 2$ *and* $0\leq c_1,\dots,c_r\leq\mathfrak{d}-1$ *be integers. Then*

$$\rho_Q^r(\mathbf{c},\mathfrak{d}) = \frac{2}{\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})}\operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big)+O\big(rQ^{-1}\log^2 Q\big)\,. \tag{10}$$

**Corollary 1.** *Let* $r\geq 3$*,* $\mathfrak{d}\geq 2$ *and* $0\leq c_1,\dots,c_r\leq\mathfrak{d}-1$ *be integers. Then, there exists the limit* $\rho^r(\mathbf{c},\mathfrak{d}):=\lim_{Q\to\infty}\rho_Q^r(\mathbf{c},\mathfrak{d})$*, and*

$$\rho^r(\mathbf{c},\mathfrak{d}) = \frac{2}{\mathfrak{d}^2}\prod_{p\mid\mathfrak{d}}\left(1-\frac{1}{p^2}\right)^{-1}\sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})}\operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big)\,.$$

## 6. On short intervals

We now turn to see what changes occur when we treat the same problem on an arbitrary interval. By the periodicity modulo an interval of length 1 of consecutive denominators of fractions in $\mathfrak{F}_Q^{\mathcal{I}}$, we may reduce to consider only shorter intervals. Thus, in the following we assume that the interval $\mathcal{I} \subseteq [0,1]$, of positive length, is fixed. In the notations introduced above for different sets, we will use an additional superscript $\mathcal{I}$ with the significance that their elements correspond to fractions from $\mathcal{I}$.

By the fundamental relation (1), we find that if $(a'/q', a''/q'') \in \mathfrak{F}_Q^2$ then

$$a''/q'' \in \mathcal{I} \iff \overline{q'} \in q''\mathcal{I}\,, \tag{11}$$

in which $\overline{q'} \in [0, q'']$ is the inverse of $q' \pmod{q''}$. Let

$$\mathfrak{G}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \big\{(q_1,\dots,q_r) \in \mathfrak{D}_Q^{\mathcal{I},r}\colon\ q_j \equiv c_j \pmod{\mathfrak{d}},\ \text{for } j=1,\dots,r\big\}$$

and $\mathbf{N}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \#\mathfrak{G}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d})$. Then, our task is to estimate

$$\rho_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) := \frac{\mathbf{N}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d})}{\#\mathfrak{F}_Q^{\mathcal{I}} - (r-1)}\,. \tag{12}$$

Now we consider the set

$$\begin{aligned}
\mathcal{G}_Q^{\mathcal{I}}(\mathbf{c},\mathfrak{d}) &= \left\{ (q',q'') \in \mathcal{T}_Q\colon \begin{array}{l} \overline{q'} \in q''\mathcal{I},\ (q',q'') \equiv (c_1,c_2) \pmod{\mathfrak{d}}, \\ \gcd(q',q'')=1,\ \mathbf{k}(q',q'') \in \mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d}) \end{array} \right\} \\
&= \bigcup_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} \left\{ (q',q'') \in Q\mathcal{T}[\mathbf{k}]\colon \begin{array}{l} \overline{q'} \in q''\mathcal{I},\ \gcd(q',q'')=1, \\ (q',q'') \equiv (c_1,c_2) \pmod{\mathfrak{d}} \end{array} \right\}.
\end{aligned}$$

Then

$$\mathbf{N}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \#\mathcal{G}_Q^{\mathcal{I}}(\mathbf{c},\mathfrak{d}) + O(1)\,. \tag{13}$$

For a given plane domain $\Omega$, we use the following notation

$$N'^{,\mathcal{I}}_{c_1,c_2;\mathfrak{d}}(\Omega) := \#\left\{ (q',q'') \in \Omega \cap \mathbb{N}^2\colon \begin{array}{l} \overline{q'} \bmod q'' \in q''\mathcal{I},\ \gcd(q',q'')=1, \\ (q_1,q_2) \equiv (c_1,c_2) \pmod{\mathfrak{d}} \end{array} \right\}.$$

Then, (13) becomes

$$\mathbf{N}_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},\mathfrak{d})} N'^{,\mathcal{I}}_{c_1,c_2;\mathfrak{d}}\big(\mathcal{T}[\mathbf{k}]\big) + O(1)\,. \tag{14}$$

The next lemma shows that $N'^{,\mathcal{I}}_{c_1,c_2;\mathfrak{d}}(\Omega)$ is, roughly, $N'_{c_1,c_2;\mathfrak{d}}(\Omega)$ times the length of the interval $\mathcal{I}$.

**Lemma 3.** *Let $Q>0$ and let $\Omega$ be a convex domain included in the triangle of vertices $(Q,0);(Q,Q);(0,Q)$. Let $\mathfrak{d}$ be a positive integer and let $0\le a,b<\mathfrak{d}$, with* $\gcd(a,b)=1$. *Then, we have*

$$N^{',\mathcal{I}}_{a,b;\mathfrak{d}}(\Omega)=|\mathcal{I}|\cdot N^{'}_{a,b;\mathfrak{d}}(\Omega)+O(Q^{3/2+\varepsilon})\,.$$

*Proof.* We count the good points in $\Omega$ situated on horizontal lines with integer coordinates. Thus, we have

$$N^{',\mathcal{I}}_{c_1,c_2;\mathfrak{d}}(\Omega)=\sum_{\substack{1\le q\le Q\\ q\equiv b \pmod{\mathfrak{d}}}}\#\left\{x\in\Omega\cap\{y=q\}:\ \begin{array}{l}\overline{x}\bmod q\in q\mathcal{I},\ \gcd(x,q)=1,\\ x\equiv a \pmod{\mathfrak{d}}\end{array}\right\}. \tag{15}$$

Employing exponential sums, the terms in the sum are equal to

$$\sum_{\substack{x\in\Omega\cap\{y=q\}\\ \gcd(x,q)=1\\ x\equiv a \pmod{\mathfrak{d}}}}\ \sum_{y\in q\mathcal{I}}\frac{1}{q}\sum_{k=1}^{q}e\Big(k\frac{y-\overline{x}}{q}\Big)=\frac{1}{q}\sum_{k=1}^{q}\sum_{y\in q\mathcal{I}}e\Big(k\frac{y}{q}\Big)\sum_{\substack{x\in\Omega\cap\{y=q\}\\ \gcd(x,q)=1\\ x\equiv a \pmod{\mathfrak{d}}}}e\Big(k\frac{-\overline{x}}{q}\Big)\,. \tag{16}$$

We separate the terms in (16) in two groups. The first one contains the terms with $k=q$ and the second is formed by all the others. The contribution of the terms from the first group will give the main term in (15), since

$$\sum_{\substack{1\le q\le Q\\ q\equiv b \pmod{\mathfrak{d}}}}\ \sum_{\substack{x\in\Omega\cap\{y=q\}\\ \gcd(x,q)=1\\ x\equiv a \pmod{\mathfrak{d}}}}\ \sum_{y\in q\mathcal{I}}\frac{1}{q}=|\mathcal{I}|\cdot N^{'}_{a,b;\mathfrak{d}}(\Omega)+O(Q)\,. \tag{17}$$

It remains to estimate the size of the terms from the second group. The second sum from the right-hand side of (16) is a geometric progression that is bounded sharply by $\ll\min(q|\mathcal{I}|,\|k/q\|^{-1})$, where $\|\cdot\|$ is the distance to the closest integer. The most inner sum is a Kloosterman-type sum. It is incomplete, both on the length of the interval and on the $x$-domain–an arithmetic progression. A standard procedure, using the classic bound of Esterman [16] and Weil [25], gives

$$\Bigg|\sum_{\substack{x\in\Omega\cap\{y=q\}\\ \gcd(x,q)=1\\ x\equiv a \pmod{\mathfrak{d}}}}e\Big(k\frac{-\overline{x}}{q}\Big)\Bigg|\le\sigma_0(q)(k,q)^{1/2}q^{1/2}(2+\log q)\,,$$

where $\sigma_l(q)$ is the sum of the $l$-th power of divisors of $q$. Thus, the contribution to (15) of the terms from the second group is

$$\ll\sum_{\substack{1\le q\le Q\\ q\equiv b \pmod{\mathfrak{d}}}}\sigma_0(q)q^{1/2}(2+\log q)\frac{1}{q}\sum_{k=1}^{q-1}(k,q)^{1/2}\min(q|\mathcal{I}|,\|k/q\|^{-1})\,. \tag{18}$$

Here the sum over $k$ is

$$\begin{aligned}\sum_{k=1}^{q}(k,q)^{1/2}\|k/q\|^{-1} &= \sum_{g|q}\sum_{\substack{k=1\\(k,q)=g}}^{q}\frac{g^{1/2}}{\|k/q\|}\\ &= \sum_{g|q} g^{1/2}\sum_{k=1}^{\left[\frac{q-1}{2g}\right]}\frac{2q}{gk}\\ &\le 2\sigma_{-1/2}(q)q(2+\log q)\,.\end{aligned}$$

On inserting this estimate in (18) and using the fact that $\sigma_l(q) \ll q^{\varepsilon}$, we see that the contribution of terms from the second group is $\ll Q^{3/2+\varepsilon}$. This completes the proof of the lemma. □

In particular, Lemma 3 may be used to count the fractions from an interval. Thus, we have

$$\#\mathfrak{F}_Q^{\mathcal{I}} = |\mathcal{I}|\cdot\#\mathfrak{F}_Q + O\big(Q^{3/2+\varepsilon}\big)\,. \tag{19}$$

Finally, using the estimate given by Lemma 3 in (14) and combining the result and (19) in (12), we obtain the following theorem.

**Theorem 3.** *Let $r \ge 1$, $\mathfrak{d} \ge 2$ and $0 \le c_1,\ldots,c_r \le \mathfrak{d}-1$ be integers. Then*

$$\rho_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \rho_Q^{r}(\mathbf{c},\mathfrak{d}) + O\big(Q^{-1/2}\log^2 Q\big)\,.$$

As a consequence, it follows that independent on the interval, the limit $\rho^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) := \lim_{Q\to\infty}\rho_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d})$ exists.

**Corollary 2.** *Let $r \ge 1$, $\mathfrak{d} \ge 2$ and $0 \le c_1,\ldots,c_r \le \mathfrak{d}-1$ be integers. Then, for any interval $\mathcal{I}$ of positive length, the sequence $\{\rho_Q^{\mathcal{I},r}(\mathbf{c},\mathfrak{d})\}_Q$ has a limit $\rho^{\mathcal{I},r}(\mathbf{c},\mathfrak{d})$, as $Q\to\infty$, and*

$$\rho^{\mathcal{I},r}(\mathbf{c},\mathfrak{d}) = \rho^{r}(\mathbf{c},\mathfrak{d})\,.$$

## 7. A few special cases

We begin with the case $\mathfrak{d} = 2$. Then $\mathbf{c} \in \{0,1\}^r$, that is we are looking to the probability that $r$-tuples of consecutive denominators are odd or even in a prescribed order. We may always suppose that there are no neighbor even components of $\mathbf{c}$, since in that case $\rho^r(\mathbf{c},2) = 0$.

When $r = 1$ or $r = 2$, we already know from the beginning of Section 5, relations (5) and (6), the precise values of $\rho^r(\mathbf{c},2)$, while for $r \ge 3$, we get $\rho^r(\mathbf{c},2)$ from Theorem 1.

**Theorem 4.** *Let $r \geq 1$ and $c_1, \dots, c_r \in \{0,1\}$. Then, there exists the limit $\rho^r(\mathbf{c},2) = \lim_{Q\to\infty} \rho^r_Q(\mathbf{c},2)$. Furthermore, we have: $\rho^1(0;2) = 1/3$, $\rho^1(1;2) = 2/3$, $\rho^1(0,1;2) = \rho^1(1,0;2) = \rho^1(1,1;2) = 1/3$, and*

$$\rho^r(\mathbf{c},2) = \frac{2}{3} \sum_{\mathbf{k}\in\mathcal{K}^{r-2}(\mathbf{c},2)} \operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big), \qquad \text{for } r \geq 3\,. \tag{20}$$

We have calculated the sums from the right-hand side of (20) in a few cases. Here they are. First we remark that when $\mathbf{k}$ has only one component (i.e. it corresponds to 3-tuples of consecutive denominators), the areas are: $\operatorname{Area}\big(\mathcal{T}[1]\big) = 1/6$ and $\operatorname{Area}\big(\mathcal{T}[k]\big) = 4/\big(k(k+1)(k+2)\big)$, for $k \geq 2$. Then, employing the sum of the Leibniz series, we obtain:

$$\begin{aligned}
\rho^3(1,1,1;2) &= \frac{2}{3} \sum_{\substack{k\geq 1\\ k \text{ even}}} \operatorname{Area}\big(\mathcal{T}[k]\big) = 2 - \frac{8}{3}\log 2 \approx 0.15160\,;\\
\rho^3(1,1,0;2) &= \frac{2}{3} \sum_{\substack{k\geq 1\\ k \text{ odd}}} \operatorname{Area}\big(\mathcal{T}[k]\big) = \frac{8}{3}\log 2 - \frac{5}{3} \approx 0.18172\,;\\
\rho^3(1,0,1;2) &= \frac{2}{3} \sum_{k\geq 1} \operatorname{Area}\big(\mathcal{T}[k]\big) = \frac{1}{3} \approx 0.33333;\\
\rho^3(0,1,1;2) &= \rho^3(1,1,0;2) = \frac{8}{3}\log 2 - \frac{5}{3} \approx 0.18172\,;\\
\rho^3(0,1,0;2) &= \frac{2}{3} \sum_{\substack{k\geq 1\\ k \text{ even}}} \operatorname{Area}\big(\mathcal{T}[k]\big) = 2 - \frac{8}{3}\log 2 \approx 0.15160\,.
\end{aligned}$$

Thus, out of the 8 possible vectors $\mathbf{c}$, only 5 are not trivial (since the others have two neighbor even denominators, so $\rho^3(0,0,1;2) = \rho^3(1,0,0;2) = \rho^3(0,0,0;2) = 0$). Furthermore two of them form a couple with the same occurring probability, because their components are mirrorly reflected of one another.

For longer sequences $\mathbf{c}$, the sum from the right-hand side of (20) involves the Leibniz series again, more precisely its smaller and smaller remainder. This happens because more and more $\mathbf{k}$'s with all components small belong to $\mathcal{K}^{r-2}(\mathbf{c},2)$, while those with at least one component $k$, say, passing over a certain magnitude have the property that $\operatorname{Area}\big(\mathcal{T}[\mathbf{k}]\big) = \operatorname{Area}\big(\mathcal{T}[k]\big)$. We mention here only that

$$\operatorname{Area}\big(\mathcal{T}[1,k]\big) = \operatorname{Area}\big(\mathcal{T}[k,1]\big) = \operatorname{Area}\big(\mathcal{T}[k]\big)\,, \qquad \text{for } k \geq 5;$$

$$\operatorname{Area}\big(\mathcal{T}[1,k,1]\big) = \operatorname{Area}\big(\mathcal{T}[k]\big)\,, \qquad \text{for } k \geq 5;$$

$$\operatorname{Area}\big(\mathcal{T}[2,1,k]\big) = \operatorname{Area}\big(\mathcal{T}[k,1,2]\big) = \operatorname{Area}\big(\mathcal{T}[k]\big)\,, \qquad \text{for } k \geq 9.$$

In the case $r = 4$, there are 8 nontrivial vectors $\mathbf{c}$, of which 5 are essentially distinct (non mirror reflected of another). Here are the probabilities with which they come about:

$$\rho^4(1,1,1,1;2) = \frac{2}{3} \sum_{\substack{k,l\geq 1 \\ k \text{ even}, \ ,l \text{ even}}} \text{Area}\big(\mathcal{T}[k,l]\big) = \frac{23}{315} \approx 0.07301\,;$$

$$\rho^4(1,1,1,0;2) = \frac{2}{3} \sum_{\substack{k,l\geq 1 \\ k \text{ even}, \ l \text{ odd}}} \text{Area}\big(\mathcal{T}[k,l]\big) = \frac{607}{315} - \frac{8}{3}\log 2 \approx 0.07859\,;$$

$$\rho^4(1,1,0,1;2) = \frac{2}{3} \sum_{\substack{k,l\geq 1 \\ k \text{ odd}}} \text{Area}\big(\mathcal{T}[k,l]\big) = \frac{8}{3}\log 2 - \frac{5}{3} \approx 0.18172\,;$$

$$\rho^4(1,0,1,1;2) = \rho^4(1,1,0,1;2) = \frac{8}{3}\log 2 - \frac{5}{3} \approx 0.18172\,;$$

$$\rho^4(1,0,1,0;2) = \frac{2}{3} \sum_{\substack{k,l\geq 1 \\ l \text{ even}}} \text{Area}\big(\mathcal{T}[k,l]\big) = 2 - \frac{8}{3}\log 2 \approx 0.15160\,;$$

$$\rho^4(0,1,1,1;2) = \rho^4(1,1,1,0;2) = \frac{607}{315} - \frac{8}{3}\log 2 \approx 0.07859\,;$$

$$\rho^4(0,1,1,0;2) = \frac{2}{3} \sum_{\substack{k,l\geq 1 \\ k \text{ odd}, \ l \text{ odd}}} \text{Area}\big(\mathcal{T}[k,l]\big) = \frac{16}{3}\log 2 - \frac{1132}{315} \approx 0.10313\,;$$

$$\rho^4(0,1,0,1;2) = \rho^4(1,0,1,0;2) = 2 - \frac{8}{3}\log 2 \approx 0.15160\,.$$

When $r = 5$, out of the 32 vectors $\mathbf{c} \in \{0,1\}^5$, only 13 have no neighbor even-even components and 9 are essentially distinct. We present the probabilities $\rho^5(\mathbf{c},2)$ in Table 1.

Table 1: The probabilities $\rho^5(\mathbf{c},2)$. In the $\mathbf{k}$-column, the notations $e$, $o$ and $\forall$ mean that the sum from the right-hand side of (20) runs over all vectors with the correspondent component even, odd, or whatever, respectively.

| $\mathbf{c}$ | $\mathbf{k}$ | $\rho^5(\mathbf{c},2)$ | approximation |
|---|---|---|---|
| $(1,1,1,1,1)$ | $(e,e,e)$ | $\frac{1}{21}$ | 0.04761 |
| $(1,1,1,1,0)$ | $(e,e,o)$ | $\frac{8}{315}$ | 0.02539 |
| $(1,1,1,0,1)$ | $(e,o,\forall)$ | $\frac{607}{315} - \frac{8}{3}\log 2$ | 0.07859 |
| $(1,1,0,1,1)$ | $(o,\forall,o)$ | $\frac{4441}{38610}$ | 0.11502 |
| $(1,1,0,1,0)$ | $(o,\forall,e)$ | $\frac{8}{3}\log 2 - \frac{68791}{38610}$ | 0.06670 |
| $(1,0,1,1,1)$ | $(\forall,o,e)$ | $\frac{607}{315} - \frac{8}{3}\log 2$ | 0.07859 |

*continued on next page*

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| $\mathbf{c}$ | $\mathbf{k}$ | $\rho^5(\mathbf{c},2)$ | approximation |
|---|---|---|---|
| $(1,0,1,1,0)$ | $(\forall,o,o)$ | $\frac{16}{3}\log 2-\frac{1132}{315}$ | 0.10313 |
| $(1,0,1,0,1)$ | $(\forall,e,\forall)$ | $2-\frac{8}{3}\log 2$ | 0.15160 |
| $(0,1,1,1,1)$ | $(o,e,e)$ | $\frac{8}{315}$ | 0.02539 |
| $(0,1,1,1,0)$ | $(o,e,o)$ | $\frac{599}{315}-\frac{8}{3}\log 2$ | 0.05319 |
| $(0,1,1,0,1)$ | $(o,o,\forall)$ | $\frac{16}{3}\log 2-\frac{1132}{315}$ | 0.10313 |
| $(0,1,0,1,1)$ | $(e,\forall,o)$ | $\frac{8}{3}\log 2-\frac{68791}{38610}$ | 0.06670 |
| $(0,1,0,1,0)$ | $(e,\forall,e)$ | $\frac{146011}{38610}-\frac{16}{3}\log 2$ | 0.08490 |

Many patterns of consecutive denominators extend without bound as $Q$ gets large. We mention here the one with all components equal modulo $\mathfrak{d}$. Let $\mathfrak{d}\ge 2$ and $0\le c\le \mathfrak{d}-1$. The condition of neighborship produces the constrain $\gcd(c,\mathfrak{d})=1$. Let $\mathbf{c}=(c,\dots,c)$ be the vector with all the $r$ components equal to $c$. Remarkably, when $r\ge 5$, there exists only one $\mathbf{k}$ which accommodates the appearing in $\mathfrak{F}_Q$ of sequences of consecutive denominators that are congruent with $\mathbf{c}$ modulo $\mathfrak{d}$. This is the vector $\mathbf{k}=(2,\dots,2)$ with $r-2$ components all equal with 2. The corresponding polygon is the quadrangle (below we refer to formulas proved in [15])

$$\mathcal{T}_{r-2}[2,\dots,2]=\Big\{(1,1);\ \Big(\frac{r-2}{2r-3},\frac{r-1}{2r-3}\Big);\ \Big(\frac{1}{2},\frac{1}{2}\Big);\ \Big(1,\frac{2r-4}{2r-3}\Big)\Big\}\,,\quad \text{for } r\ge 3$$

and its area is

$$\operatorname{Area}\big(\mathcal{T}_{r-2}[2,\dots,2]\big)=\frac{1}{2(2r-3)}\,,\quad \text{for } r\ge 3.$$

(Since $r$, the number of components of $\mathbf{k}$ is essential in the formulae, in order to indicate precisely its size, we write $\mathcal{T}_r[\mathbf{k}]$ instead of $\mathcal{T}[\mathbf{k}]$.) Then, Corollary 1 yields the following result.

**Corollary 3.** *Let $r\ge 5$, $\mathfrak{d}\ge 2$, and let $0\le c\le\mathfrak{d}-1$ with $\gcd(c,\mathfrak{d})=1$. Then*

$$\rho^r(c,\dots,c;\mathfrak{d})=\frac{1}{\mathfrak{d}^2(2r-3)}\prod_{p\mid\mathfrak{d}}\Big(1-\frac{1}{p^2}\Big)^{-1}.$$

In particular, Corollary 3 gives the probability to find $r\ge 5$ odd consecutive denominators:

$$\rho^r(\underbrace{1,\dots,1}_{r\text{ ones}};2)=\frac{1}{3(2r-3)}\,,\quad \text{for } r\ge 5\,. \tag{21}$$

The same pattern boarded on either side by an even denominator is very similar. Indeed, if $\mathbf{c} = (0, \underbrace{1, \ldots, 1}_{r-1 \text{ ones}})$ and $r \geq 6$, there exists only two corresponding vectors: $\mathbf{k} = (1, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}})$ and $\mathbf{k} = (3, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}})$, while the mirror reflected case $\mathbf{c} = (\underbrace{1, \ldots, 1}_{r-1 \text{ ones}}, 0)$ corresponds to $\mathbf{k} = (\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 1)$ and $\mathbf{k} = (\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 3)$. In all four cases $\mathcal{T}[\mathbf{k}]$ is a triangle:

$$\mathcal{T}[1, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}}] = \Big\{(0,1);\ \Big(\frac{1}{2r-3}, \frac{2r-4}{2r-3}\Big);\ \Big(\frac{1}{2r-5}, 1\Big)\Big\}, \quad \text{for } r \geq 3\,;$$

$$\mathcal{T}[3, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}}] = \Big\{\Big(1, \frac{1}{2}\Big);\ \Big(1, \frac{r-1}{2r-3}\Big);\ \Big(\frac{2r-7}{2r-5}, \frac{r-3}{2r-5}\Big)\Big\}, \quad \text{for } r \geq 5\,;$$

$$\mathcal{T}[\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 1] = \Big\{(1,1);\ \Big(\frac{r-3}{2r-5}, \frac{r-2}{2r-5}\Big);\ \Big(\frac{r-2}{2r-3}, \frac{r-1}{2r-3}\Big)\Big\}, \quad \text{for } r \geq 3\,;$$

$$\mathcal{T}[\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 3] = \Big\{\Big(\frac{1}{2}, \frac{1}{2}\Big);\ \Big(1, \frac{2r-6}{2r-5}\Big);\ \Big(1, \frac{2r-4}{2r-3}\Big)\Big\}, \quad \text{for } r \geq 5\,;$$

and they have the same area:

$$\begin{aligned}
\operatorname{Area}\big(\mathcal{T}[1, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}}]\big) &= \operatorname{Area}\big(\mathcal{T}[\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 1]\big)\\
&= \operatorname{Area}\big(\mathcal{T}[3, \underbrace{2, \ldots, 2}_{r-3 \text{ twos}}]\big) = \operatorname{Area}\big(\mathcal{T}[\underbrace{2, \ldots, 2}_{r-3 \text{ twos}}, 3]\big)\\
&= \frac{1}{2(2r-5)(2r-3)}\,, \quad \text{for } r \geq 5\,.
\end{aligned}$$

Then, by Corollary 1, we get

$$\rho^r(0, \underbrace{1, \ldots, 1}_{r-1 \text{ ones}}; 2) = \rho^r(\underbrace{1, \ldots, 1}_{r-1 \text{ ones}}, 0; 2) = \frac{2}{3(2r-5)(2r-3)}\,, \quad \text{for } r \geq 6\,.$$

We conclude with an analogue example on the side of $\mathbf{k}$'s. The question we address is whether beside $\mathbf{k} = (2, \ldots, 2)$, there exists another $\mathbf{k}$ with all components equal, which extends without bound. This happens, but only modulo some $\mathfrak{d} \geq 2$, namely $\mathfrak{d} = 3$, and $\mathbf{k}$ being a series of ones intercalated by fours. As $\mathbf{k}$ and its pal, the one with components in reversed order, satisfy the demands at the same time, the components of these vectors

depends on the parity of $r$. Precisely, for any $r \geq 1$, they are:

$$\mathcal{T}_{2r}[1,4,\ldots,1,4] = \Big\{\Big(\frac{1}{3},\frac{2}{3}\Big);\Big(\frac{3r}{6r-1},1\Big);\Big(\frac{1}{2},1\Big);\Big(\frac{2r}{6r+1},\frac{4r+1}{6r+1}\Big)\Big\};$$
$$\mathcal{T}_{2r+1}[1,4,\ldots,4,1] = \Big\{\Big(\frac{1}{3},\frac{2}{3}\Big);\Big(\frac{3r+1}{6r+1},1\Big);\Big(\frac{1}{2},1\Big);\Big(\frac{2r}{6r+1},\frac{4r+1}{6r+1}\Big)\Big\};$$
$$\mathcal{T}_{2r}[4,1\ldots,4,1] = \Big\{\Big(1,\frac{1}{2}\Big);\Big(\frac{4r-1}{6r-1},\frac{2r}{6r-1}\Big);\Big(\frac{2}{3},\frac{1}{3}\Big);\Big(1,\frac{3r}{6r+1}\Big)\Big\};$$
$$\mathcal{T}_{2r+1}[4,1,\ldots,1,4] = \Big\{\Big(1,\frac{1}{2}\Big);\Big(\frac{4r+3}{6r+5},\frac{2r+2}{6r+5}\Big);\Big(\frac{2}{3},\frac{1}{3}\Big);\Big(1,\frac{3r+2}{6r+5}\Big)\Big\}.$$

And here are their areas:

$$\begin{aligned}
\operatorname{Area}\big(\mathcal{T}_{2r}[1,4,\ldots,1,4]\big) &= \frac{r}{36r^2-1};\\
\operatorname{Area}\big(\mathcal{T}_{2r+1}[1,4,\ldots,4,1]\big) &= \frac{1}{36r+6};\\
\operatorname{Area}\big(\mathcal{T}_{2r}[4,1\ldots,4,1]\big) &= \frac{r}{36r^2-1};\\
\operatorname{Area}\big(\mathcal{T}_{2r+1}[4,1,\ldots,1,4]\big) &= \frac{1}{36r+30}.
\end{aligned} \tag{22}$$

One of the patterns suited by this $\mathbf{k}$'s is formed by sequences of denominators that are congruent modulo 3 with a series of repeated ones and twos. For these, we obtain the following probabilities.

**Corollary 4.** *For any $r \geq 4$, we have:*

$$\rho^{2r}(1,2,\ldots,1,2;3) = \rho^{2r}(1,2,\ldots,1,2;3) = \frac{r-1}{72(r-1)^2-2},$$
$$\rho^{2r+1}(1,2,\ldots,2,1;3) = \rho^{2r+1}(2,1,\ldots,1,2;3) = \frac{9r+4}{8(9r+1)(9r+7)}.$$

*Proof.* By Corollary 1, we have:

$$\rho^{2r}(1,2,\ldots,1,2;3) = \frac{1}{4}\Big(\operatorname{Area}\big(\mathcal{T}_{2(r-1)}[1,4,\ldots,1,4]\big) + \operatorname{Area}\big(\mathcal{T}_{2(r-1)}[4,1,\ldots,4,1]\big)\Big),$$
$$\rho^{2r+1}(1,2,\ldots,2,1;3) = \frac{1}{4}\Big(\operatorname{Area}\big(\mathcal{T}_{2r-1}[1,4,\ldots,4,1]\big) + \operatorname{Area}\big(\mathcal{T}_{2r-1}[4,1,\ldots,1,4]\big)\Big),$$

and the same relations apply for $\rho^{2r}(2,1,\ldots,2,1;3)$ and $\rho^{2r+1}(2,1,\ldots,1,2;3)$, respectively. Then the corollary follows using the formulae from (22). □

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CC: Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 70700, Romania.

*E-mail address*: cristian.cobeli@imar.ro

AZ: Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 70700, Romania.

*Current address*: AZ: Department of Mathematics, University of Illinois at Urbana-Champaign, Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA.

*E-mail address*: zaharesc@math.uiuc.edu